ANALYSIS OF SHAKEDOWN OF NONUNIFORMLY HEATED ELASTOPLASTIC BODIES

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Well-known theorems of Melan and Koiter (see, for example, [1]) are of fundamental importance in shakedown theory; their combined use permits in principle a two-sided estimate of the shakedown loads. It should be pointed out that as yet only Melan's theorem has been used in practice.

Variable conditions due to the presence of nonsteady temperature fields are of considerable interest in several branches of technology. The shakedown of nonuniformly heated bodies and the corresponding generalization of Melan's theorem were considered in [2–6] and other papers. In the present investigation we give an analogous generalization of Koiter's theorem and consider some possible applications of an approximate analysis based on this theorem.

1. We assume that an elastic-perfectly plastic body is subjected to the action of several loads, varying in a certain range, and a nonsteady temperature Θ (x, y, z, t), which varies at each point between certain specific values. We denote by $\sigma_{ij}{}^e(t)$, $\varepsilon_{ij}{}^e(t)$ the (fictitious) thermoelastic solution corresponding to ideally elastic behavior. The values $\sigma_{ij}{}^e(t)$, $\varepsilon_{ij}{}^e(t)$ are linked by the relation

$$\varepsilon_{ij}^e = C_{ijhk}\sigma_{hk}^e + \delta_{ij}\alpha\theta, \tag{1.1}$$

where C_{ijhk} is the elasticity tensor ($C_{ijhk} = C_{hkij}$), and δ_{ij} is the Kronecker symbol.

The true (elastoplastic) stresses and strains at any moment of time can be represented by

$$\sigma_{ij} = \sigma_{ij}^e + \rho_{ij}, \qquad \varepsilon_{ij} = \varepsilon_{ij}^e + e_{ij}. \tag{1.2}$$

Here ρ_{ij}, e_{ij} are the (instantaneous) residual stresses and strains. The total strain ε_{ij} can also be represented in the form

$$\varepsilon_{ij} = \varepsilon_{ij}' + \varepsilon_{ij}'' \qquad (\varepsilon_{ij}' = C_{ijhk}\sigma_{hk} + \delta_{ij}\alpha\theta). \tag{1.3}$$

Here ε_{ij}', ε_{ij}'' are the elastic and plastic components, respectively.

By combining relations (1.1)–(1.3) it is easy to see that the residual stresses and strains satisfy the relation

$$e_{ij} = C_{ijhk}\rho_{hk} + \varepsilon_{ij}''. \tag{1.4}$$

All the introduced stress and strain tensors are slowly changing functions of time. Stresses σ_{ij} and $\sigma_{ij}{}^e$ are statically possible fields balanced by instantaneous external surface and volume loads; the residual stresses ρ_{ij} form a self-equilibrated field, while strains ε_{ij}, $\varepsilon_{ij}{}^e$, e_{ij} are kinematically possible, i.e., they are compatible, the corresponding displacement fields satisfying the assumed kinematic boundary conditions at the surface;[1] finally, the components ε_{ij}'' and ε_{ij}' are not compatible.

The following basic inequality [1] applies for plastic flow (local maximum principle):

$$(\sigma_{ij} - \sigma_{ij}^*)\dot{\varepsilon}_{ij}'' > 0 \tag{1.5}$$

where $\sigma_{ij}{}^*$ is an arbitrary state of stress inside the yield surface (the dot denotes differentiation with respect to time).

Along with the actual plastic strains $\varepsilon_{ij}''(t)$ we shall also consider a certain arbitrary field of plastic strains $\varepsilon_{ij_0}''(t)$, which we shall call "admissible" if the plastic strain increments

$$\Delta\varepsilon_{ij_0}'' = \int_0^T \dot{\varepsilon}_{ij_0}'' dt \tag{1.6}$$

calculated for a certain interval of time T, form a kinematically possible field. After setting $\varepsilon_{ij}'' = \varepsilon_{ij_0}''$ in (1.4), we find a certain (unique) distribution of "associated" residual stresses ρ_{ij_0}, residual strains e_{ij_0} and residual displacements u_{ij_0}.[1] It should be noted that, owing to the condition (1.6) and relation (1.4), at the end of a cycle the associated residual stresses return to the values which they had at the beginning of the cycle

$$\rho_{ij_0}|_{t=0} = \rho_{ij_0}|_{t=T}. \tag{1.7}$$

2. The Koiter theorem for a nonuniformly heated body can be formulated as follows: no shakedown takes place if an admissible cycle of plastic strain rates $\dot{\varepsilon}_{ij_0}''(t)$ and a certain program of load and temperature variation (within given limits) can be found for which

$$\int_0^T dt \left\{\int_v F_i \dot{u}_{i_0} dv + \int_{S_p} p_i \dot{u}_{i_0} ds + \int_v \alpha\theta\dot{\rho}_{ii_0} dv\right\} > \int_0^T dt \int_v W(\dot{\varepsilon}_{ij_0}'')\, dt. \tag{2.1}$$

On the other hand, the system shakes down, if for all admissible cycles of plastic strain rates and all possible variations of load and temperature (within given limits), relation (2.1) is satisfied with the inequality sign reversed. In (2.1) $W(\dot{\varepsilon}_{ij_0}'')$ denotes the rate of plastic dissipation of energy at admissible strain rates $\dot{\varepsilon}_{ij_0}''(t)$.

In order to prove the first part of the theorem, we assume, following [1], that although there exists a cycle which satisfies (2.1) shakedown is observed. Then, according to Melan's theorem for a nonuniformly heated body [2, 4], it is possible to find a steady field of residual stresses $\rho_{ij}{}^\circ$ such that the sun

$$\sigma_{ij}^e + \rho_{ij}^\circ = \sigma_{ij}^* \tag{2.2}$$

nowhere exceeds the yield stress. According to the principle of virtual work

$$\int_v F_i \dot{u}_{i_0} dv + \int_{S_p} p_i \dot{u}_{i_0} dS = \int_v \sigma_{ij}^* \dot{e}_{ij_0} dv. \tag{2.3}$$

Since, according to (1.4),

$$\dot{e}_{ij_0} = C_{ijhk}\dot{\rho}_{hk_0} + \dot{\varepsilon}_{ij_0}'', \tag{2.4}$$

we can represent (using (2.2)) the right-hand side of (2.3) in the following form:

$$\int_v \sigma_{ij}^* \dot{e}_{ij_0} dv = \int_v \sigma_{ij}^e C_{ijhk}\dot{\rho}_{hk_0} dv + \int_v \rho_{ij}^\circ C_{ijhk}\dot{\rho}_{hk_0} dv + \int_v \sigma_{ij}^* \dot{\varepsilon}_{ij_0}'' dv. \tag{2.5}$$

The first term of the right-hand side can be transformed by means of (1.1) as follows:

$$\int_v \sigma_{ij}^e C_{ijhk}\dot{\rho}_{hk_0} dv = \int_v \varepsilon_{ij}^e \dot{\rho}_{ij_0} dv - \int_v \alpha\theta\delta_{ij}\dot{\rho}_{ij_0} dv. \tag{2.6}$$

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[1]We shall consider only the case when the kinematic boundary conditions (if these conditions are given) correspond to the vanishing of certain components of the displacement vector.

[1]In order actually to connect these fields it is obviously necessary to solve the elastic boundary problem for zero loads and given (zero) displacements at the surface in the presence of "superposed" strains ε_{ij}''.

Because of the principle of virtual work the first term in the right-hand side is zero. If we now integrate relation (2.5) with respect to time, from t = 0 to t = T, because of (1.2) the second term of the right-hand side of (2.5) vanishes. As a result, we obtain the relation

$$\int_0^T dt\left\{\int_v F_i u_{i_0}^{\cdot}\,dv + \int_{S_p} p_i u_{i_0}^{\cdot}\,ds + \int_v \alpha\theta\dot\rho_{ii_0}\,dv\right\} = \int_0^T dt \int_v \sigma_{ij}^{*}\,\dot\varepsilon_{ij_0}''\,dv. \qquad (2.7)$$

The contradiction between (2.1) and (2.7), which can be easily detected by means of the inequality (1.5), proves the first part of the theorem. The second part of the theorem is proved by means of an analogous generalization of Koiter's proof.

3. The following results should be noted: steady temperature fields have no effect on adaptability. In fact, if Θ is independent of time, then the temperature term in relation (2.1)

$$\int_0^T dt \int_v \alpha\theta\dot\rho_{ii_0}\,dv = \int_v \alpha\theta\,[\rho_{ii_0}(t=T) - \rho_{ii_0}(t=0)]\,dv$$

and vanishes owing to the cyclicity condition (1.7). It is obvious that adaptation is also independent of the initial stresses or, generally, of any self-equilibrated internal steady stresses.

4. As emphasized in [1], shakedown calculations based on both Melan's theorem and the theorem of Koiter involve a very detailed elastic analysis, which makes the solution much more difficult than, for example, in the case of a similar use of the theorems of limit analysis. In particular, if the Koiter theorem is used, an original elastic problem arises in connection with the need to construct, on the basis of (2.4), the residual stresses and strains associated with a specific plastic cycle $\varepsilon^{\cdot\prime\prime}_{ij_0}(t)$. In order to eliminate this difficulty we suggest the following technique. We specify at the beginning the kinematically possible field of residual rates $u^{\cdot}_{i0}$ and the statically possible field of residual stress rates $\rho_{ij}{}^{\cdot}{}_0$. This automatically determines the corresponding admissible cycle of plastic strain rates $\varepsilon_{ij}{}''{}_0(t)$ which, according to (2.4) have the following form:

$$\dot\varepsilon_{ij_0}'' = \tfrac{1}{2}(\dot u_{i_0,j} + \dot u_{j_0,i}) - c_{ijhk}\dot\rho_{hk_0}. \qquad (4.1)$$

It is obvious that the obtained distributions of u_{i_0}, ρ_{ij_0}, ε_{ij_0} satisfy the necessary conditions and can be used for calculations from (2.1).

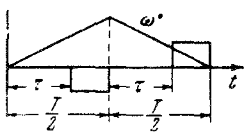

Fig. 1

5. In order to illustrate the described method we shall consider the following problem. A thin plate of any plan shape is clamped along the edges and subjected to the variable temperature field

$$\theta = \Phi(t) + \frac{z}{\tfrac{1}{2}h}\Psi(t), \qquad (5.1)$$

where h is the constant thickness of the plate, z is measured from the middle surface along the normal, and functions $\Phi(t)$ and $\Psi(t)$ can vary arbitrarily and independently of each other in the range

$$-\Phi_1 \leqslant \Phi(t) \leqslant \Phi_2, \qquad -\Psi_1 \leqslant \Psi(t) \leqslant \Psi_2. \qquad (5.2)$$

Without loss of generality, we can, according to the corollary of Section 3, restrict ourselves to the consideration of a somewhat simpler case

$$0 \leqslant \Phi(t) \leqslant \Phi_0, \qquad 0 \leqslant \Psi(t) \leqslant \Psi_0, \qquad (5.3)$$

which can be reduced to the former case by superimposing a certain steady temperature field. We begin by selecting the statically possible associated residual stresses, which we take in the following form:

$$\dot\rho_{x_0} = \dot\rho_{y_0} = -\frac{E}{1-\nu}\mu^{\cdot}(t)\,s(z), \qquad (5.4)$$

where (Fig. 1)

$$\mu^{\cdot} = 0 \ \text{for}\ 0 \leqslant t \leqslant \tau, \quad \mu^{\cdot} = 0 \ \text{for}\ \tfrac{1}{2}T \leqslant t \leqslant \tau + \tfrac{1}{2}T,$$
$$\mu^{\cdot} = -1 \ \text{for}\ \tau \leqslant t \leqslant \tfrac{1}{2}T, \quad \mu^{\cdot} = 1 \ \text{for}\ \tau + \tfrac{1}{2}T \leqslant t \leqslant T, \qquad (5.5)$$

and also

$$s(z) = 1 \quad \text{for}\ z^* \leqslant z \leqslant \tfrac{1}{2}h,$$
$$s(z) = 0 \quad \text{for}\ -\tfrac{1}{2}h \leqslant z \leqslant z^*. \qquad (5.6)$$

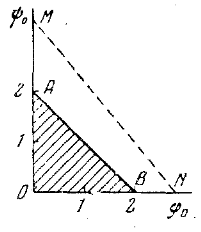

Fig. 2

Here z^* and τ are arbitrary contacts, which satisfy the conditions

$$-\tfrac{1}{2}h \leqslant z^* \leqslant \tfrac{1}{2}h, \qquad 0 \leqslant \tau \leqslant \tfrac{1}{2}T. \qquad (5.7)$$

The simplest kinematically possible field of velocities $u_i{}^{\cdot}{}_0$ vanishing at the edges is

$$\dot u_{x_0} = \dot u_{y_0} = 0.$$

The residual strain rates are also identically zero and, consequently, according to (4.1) (for an isotropic material) we obtain

$$\dot\varepsilon_{x_0}'' = \dot\varepsilon_{y_0}'' = \mu^{\cdot}(t)\,s(z). \qquad (5.8)$$

It now remains to specify the cycle of temperature variation, which we take in the form

$$\Phi(t) = \Phi_0\,\omega(t), \qquad \Psi(t) = \Psi_0\,\omega(t), \qquad (5.9)$$

where

$$\omega(t) = \frac{t}{\tfrac{1}{2}T}\ \left(0 \leqslant t \leqslant \frac{T}{2}\right), \qquad \omega(t) = 2 - \frac{t}{\tfrac{1}{2}T}\ \left(\frac{T}{2} \leqslant t \leqslant T\right). \qquad (5.10)$$

Substituting (5.4), (5.8), and (5.9) into Koiter's equation (2.1) (replacing the inequality sign by the equals sign) we obtain, after some calculations,

$$\varphi_0 + \psi_0\,\frac{1+\zeta^*}{2} = 2\,\frac{\tfrac{1}{2}T}{\tau}. \qquad (5.11)$$

Here

$$\varphi_0 = \frac{E\alpha}{(1-\nu)\sigma_s}\Phi_0, \qquad \psi_0 = \frac{E\alpha}{(1-\nu)\sigma_s}\Psi_0, \qquad \zeta^* = \frac{z^*}{\tfrac{1}{2}h}.$$

We now introduce the plane of variables $\varphi_0\Psi_0$. The straight line (5.11) defines on this plane a certain triangular region OMN of shakedown loads (Fig. 2). Since the solution (5.11) is an upper bound, the values of the parameters ζ^* and τ must be selected so as to give the lowest position of this boundary. Keeping the constraints (5.7) in mind, we obtain

$$\zeta^* = 1, \qquad \tau = \tfrac{1}{2}T. \qquad (5.12)$$

Here condition (5.11) takes the form $\varphi_0 + \Psi_0 = 2$ (straight line AB in Fig. 2).

6. In order to assess the accuracy of the solution obtained we will determine from Melan's theorem the lower bound of the adaptive load. For this purpose we must determine the stationary field of residual stresses which, when superposed on the thermoelastic solution, would result in purely elastic behavior.

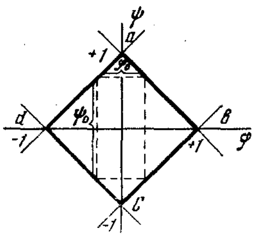

Fig. 3

The thermoelastic solution has the form

$$\sigma_x^e = \sigma_y^e = -\frac{E\alpha}{1-\nu}\left[\Phi(t) + \frac{z}{{}^1/_2 h}\Psi(t)\right]. \qquad (6.1)$$

Using the fact that stresses (6.1) are self-equilibrated, we take the residual stresses in a form similar to (6.1)

$$\rho_x^\circ = \rho_y^\circ = -\frac{E\alpha}{1-\nu}\left[M + \frac{z}{{}^1/_2 h}N\right] \qquad (M,\ N = \text{const}).$$

After introducing the total stresses $\sigma_x^e + \rho_x^o$, $\sigma_y^e + \rho_y^o$ into the Mises (or Tresca) yield condition, we obtain

$$\varphi + \zeta\psi = \pm 1 \quad \left(\varphi = \frac{E\alpha}{(1-\nu)\sigma_s}(\Phi - M),\right.$$

$$\left.\psi = \frac{E\alpha}{(1-\nu)\sigma_s}(\Psi - N),\ \zeta = \frac{z}{{}^1/_2 h}\right). \qquad (6.3)$$

On the plane $\varphi\Psi$ Eq. (6.3) defines, for all possible values of $\zeta(-1 \leq \zeta \leq 1)$, two single-parameter families of straight lines which pass through the points $\Psi = 0$, $\varphi = \pm 1$ (Fig. 3).

The outer lines of these families form the square abcd. The conditions (5.3) define on the plane $\varphi\Psi$ a certain rectangular region. It is obvious that shakedown is observed if this region is inscribed in the square abcd. From this condition we find

$$\psi_0 + \psi_0 = 2. \qquad (6.4)$$

The lower bound (6.4) thus obtained coincided with the upper bound (5.12); therefore the solution is exact.

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